Galois Groups of Purely Lacunary Polynomial Systems

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Workshop on Computational Geometry, BIRS
June 2024

Galois Groups of Enumerative Problems

An enumerative problem consists of:

- A parameter space $\mathcal{P}$.
- A solution space $\mathcal{S}$.
- An incidence space $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{S}$.

We will assume each of these spaces are algebraic varieties.

When an enumerative problem has finitely many smooth solutions for general parameters, the projection

$$\begin{array}{c} \mathcal{I} \subseteq \mathcal{P} \times \mathcal{S} \\ \pi \downarrow \quad \\ \mathcal{P} \end{array}$$

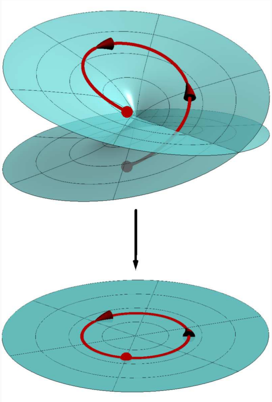
restricts to a covering space over a Zariski open set.

Galois Groups of Enumerative Problems

The Galois group of an enumerative problem is monodromy group of the projection $\pi : \mathcal{I} \to \mathcal{P}$.

- Elements are permutations of a general fiber obtained by lifting based loops.

- By ordering the fiber, the Galois group is a subgroup of the symmetric group.



"Why Galois Groups?"

- The Galois group of an enumerative problem controls the ability to symbolically compute solutions in radicals.
- Partial knowledge of the Galois group can be used to reduce computation.
- Galois groups have been useful for analyzing problems in applications.
- They were objects of interest to Galois, Jordan, Hermite, Harris, and others.

Galois Groups of Enumerative Problems

We'll consider Galois groups of sparse polynomial systems.

- ('18) Esterov studied Galois groups of sparse systems to determine which sparse systems were solvable by radicals.
- ('20) Brysiewicz, Rodriguez, Sottile, and Y. wrote software exploiting Galois groups of sparse systems for solving.
- ('22) Brysiewicz and Burr utilized Galois groups of sparse systems in creating a sparse trace test.

Sparse Polynomial Systems

Sparse polynomial systems are polynomial systems whose monomial structure has been predetermined.

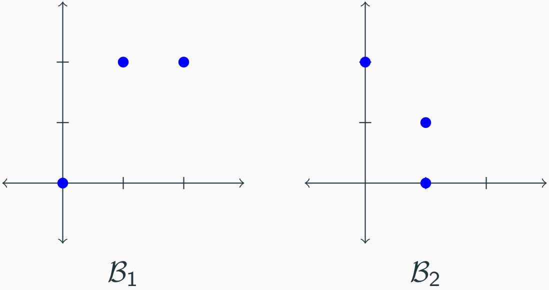
- A (Laurent) monomial with exponent vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ is $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.
- A (Laurent) polynomial f of support $\mathcal{A} \subseteq \mathbb{Z}^n$ has the form $f(x) = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha, \quad c_\alpha \in \mathbb{C}$.
- A sparse system of support $\mathcal{A}_\bullet = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is a system

$$F = (f_1, \ldots, f_n)$$

  where f_i has support $\mathcal{A}_i$ for each $i = 1, \ldots, n$.

Sparse Polynomial Systems

Example: Consider the supports $\mathcal{B}_\bullet = (\mathcal{B}_1, \mathcal{B}_2)$ depicted below.



A system F of support $\mathcal{B}_\bullet$ has the form

$$F(x,y) = \begin{pmatrix} c_1 + c_2xy^2 + c_3x^2y^2 \\ c_4x + c_5xy + c_6y^2 \end{pmatrix}.$$

The number of zeros of a general system of support $\mathcal{A}_\bullet$ is determined by the polyhedral structure of the supports.

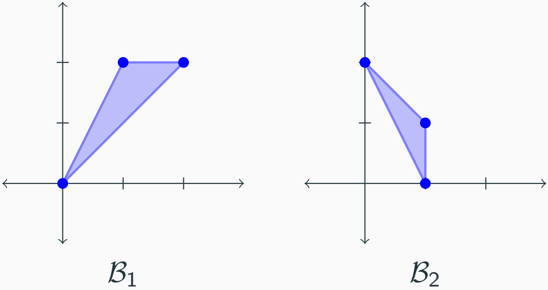
The mixed volume $\mathrm{MV}(\mathcal{C}_1, \ldots, \mathcal{C}_n)$ of a set of convex bodies $\mathcal{C}_1, \ldots, \mathcal{C}_n \subseteq \mathbb{R}^n$ is a measure of the size of the these sets.

- We write $\mathrm{MV}(\mathcal{A}_\bullet) = \mathrm{MV}(\mathrm{conv}(\mathcal{A}_1), \ldots, \mathrm{conv}(\mathcal{A}_n))$

Theorem (Bernstein,Kushnirenko,Khovanskii)
A sparse polynomial system F of support $\mathcal{A}_\bullet$ has at most $\mathrm{MV}(\mathcal{A}_\bullet)$ smooth, isolated zeros in $(\mathbb{C}^\times)^n$, and this bound is attained for a general system of support $\mathcal{A}_\bullet$.

Sparse Polynomial Systems

Example: For the support $\mathcal{B}_\bullet$ in the previous example, $\mathrm{MV}(\mathcal{B}_\bullet) = 6$.



Thus, for a general choice of coefficients $c_1, \ldots, c_6 \in \mathbb{C}$, the system

$$\begin{aligned} c_1 + c_2 x y^2 + c_3 x^2 y^2 &= 0 \\ c_4 x + c_5 x y + c_6 y^2 &= 0 \end{aligned}$$

has 6 smooth, isolated zeros in $(\mathbb{C}^\times)^2$.

Galois Groups of Sparse Polynomial Systems

Given supports $\mathcal{A}_\bullet$, we have:

- A parameter space $\mathcal{P}_{\mathcal{A}_\bullet} = \mathbb{C}^{\sum_i |\mathcal{A}_i|}$.
- A solution space $\mathcal{S} = (\mathbb{C}^\times)^n$.
- An incidence correspondence

$$\begin{array}{c} \mathcal{I}_{\mathcal{A}_\bullet} = \{(F, x) \in \mathcal{P}_{\mathcal{A}_\bullet} \times \mathcal{S} : F(x) = 0\} \\ \pi \downarrow \\ \mathcal{P}_{\mathcal{A}_\bullet} \end{array}$$

- By the BKK theorem, π restricts to a $\mathrm{MV}(\mathcal{A}_\bullet)$-sheeted covering space over a Zariski open set.

The Galois group $\mathcal{G}_{\mathcal{A}_\bullet}$ of the family of sparse polynomial systems of support $\mathcal{A}_\bullet$ is the Galois group of this enumerative problem.

Galois Groups of Sparse Polynomial Systems

Galois groups can be approximated using numerical homotopy continuation software such as `NAG4M2`, `Bertini`, and `HomotopyContinuation.jl`.

Example: Consider our running example support $\mathcal{B}_\bullet$. A system of support $\mathcal{B}_\bullet$ has the form

$$F(x,y) = \begin{pmatrix} c_1 + c_2xy^2 + c_3x^2y^2 \\ c_4x + c_5xy + c_6y^2 \end{pmatrix} = 0.$$

Tracking the zeros of a base system along various loops in $\mathcal{P}_{\mathcal{B}_\bullet}$ yields permutations which generate the symmetric group $\mathcal{S}_6$.

Galois Groups of Sparse Polynomial Systems

Open Problem: The inverse Galois problem for sparse polynomial systems: what are the groups that appear as the Galois group of a sparse polynomial system?

Open Problem: Given a sparse polynomial system, determine its Galois group.

- Esterov determined the supports $\mathcal{A}_\bullet$ for which the Galois group $\mathcal{G}_{\mathcal{A}_\bullet}$ is the symmetric group.
- Esterov's result relies on two special classes of supports.

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Galois Groups of Sparse Polynomial Systems

$\mathcal{A}_\bullet$ is lacunary if every system $F \in \mathcal{P}_{\mathcal{A}_\bullet}$ has been precomposed with a non-invertible surjective monomial map $\phi : (\mathbb{C}^\times)^n \to (\mathbb{C}^\times)^n$.

- This notion generalizes systems of the form $f(x^3) = 0$.

$\mathcal{A}_\bullet$ is triangular if every sparse system F of support $\mathcal{A}_\bullet$ has a proper, nontrivial subsystem.

- This notion generalizes systems of the form $f(x, y) = g(y) = 0$.

Theorem (Esterov)
If $\mathcal{A}_\bullet$ is not lacunary and not triangular, then the Galois group $\mathcal{G}_{\mathcal{A}_\bullet}$ is the symmetric group.

The standard argument:

- A small loop around the discriminant lifts to a simple transposition in $\mathcal{G}_{\mathcal{A}_\bullet}$.
- The Galois group $\mathcal{G}_{\mathcal{A}_\bullet}$ is 2-transitive.

We will examine the Galois group $\mathcal{G}_{\mathcal{A}_\bullet}$ when $\mathcal{A}_\bullet$ is lacunary.

Galois Groups of Sparse Polynomial Systems

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Galois Groups of Lacunary Sparse Polynomial Systems

If $\mathcal{A}_\bullet$ is lacunary, there is a support $\mathcal{B}_\bullet$ and monomial map $\phi : (\mathbb{C}^\times)^n \to (\mathbb{C}^\times)^n$ such that for every $F \in \mathcal{P}_{\mathcal{A}_\bullet}$,

$$F = G \circ \phi$$

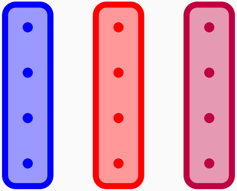
for $G \in \mathcal{P}_{\mathcal{B}_\bullet}$ with the same coefficients.

- The support $\mathcal{B}_\bullet$ is a reduced support for $\mathcal{A}_\bullet$. (We will always assume $\mathrm{MV}(\mathcal{B}_\bullet) > 1$.)
- The kernel $\mathcal{K} = \ker \phi$ is a finite group which acts on the zeros of F by coordinate-wise multiplication.

Galois Groups of Sparse Polynomial Systems

Fix a general base system $F \in \mathcal{P}_{\mathcal{A}_\bullet}$.

- The zeros of F are partitioned into $\mathcal{K}$-orbits.

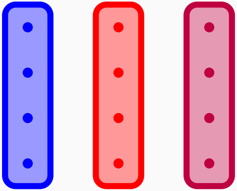


- The number of orbits is equal to $m = \mathrm{MV}(\mathcal{B}_\bullet)$ and the size of each orbit is $|\mathcal{K}|$.
- The action of $\mathcal{G}_{\mathcal{A}_\bullet}$ commutes with the action of $\mathcal{K}$ and preserves these orbits.

That is, $\mathcal{G}_{\mathcal{A}_\bullet}$ is imprimitive and the orbits form blocks of imprimitivity.

Galois Groups of Sparse Polynomial Systems

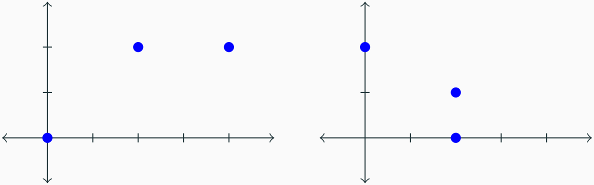
The wreath product $\mathcal{K} \wr \mathcal{S}_m$ consists of permutations that permute these blocks and in each block act by an element of $\mathcal{K}$.



- The Galois group $\mathcal{G}_{\mathcal{A}_\bullet}$ is a subgroup of the wreath product $\mathcal{K} \wr \mathcal{S}_m$.

One may *expect* that the Galois group $\mathcal{G}_{\mathcal{A}_\bullet}$ is equal to the wreath product $\mathcal{K} \wr \mathcal{S}_m$, but this is not necessarily the case!

Example: Consider the lacunary support $\mathcal{A}_\bullet$. The support $\mathcal{B}_\bullet$ is a reduced support via the monomial map $\phi(x, y) = (x^2, y)$.



$\mathcal{A}_1$ $\mathcal{A}_2$

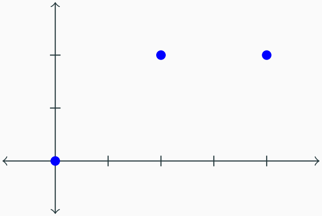
- The kernel $\mathcal{K} = \ker\phi = \{(1, 1), (-1, 1)\}$ partitions the zeros of a system $F \in \mathcal{P}_{\mathcal{A}_\bullet}$ into 6 orbits of size 2.
- The Galois group $\mathcal{G}_{\mathcal{A}_\bullet}$ is a proper subgroup of the wreath product $\mathcal{K} \wr \mathcal{S}_6$ of index 2.

Purely Lacunary Systems

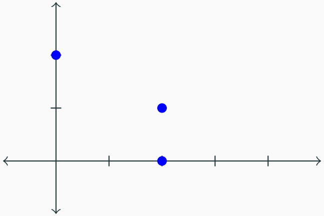
The support $\mathcal{A}_\bullet$ is purely lacunary if it is lacunary and not triangular.

- Equivalently, $\mathcal{A}_\bullet$ is purely lacunary if there is a reduced support $\mathcal{B}_\bullet$ where $\mathcal{G}_{\mathcal{B}_\bullet}$ is a symmetric group.

Example: The support $\mathcal{A}_\bullet$ below is purely lacunary–our example support $\mathcal{B}_\bullet$ is a reduced support, where $\mathcal{G}_{\mathcal{B}_\bullet} = \mathcal{S}_6$.



$\mathcal{A}_1$



$\mathcal{A}_2$

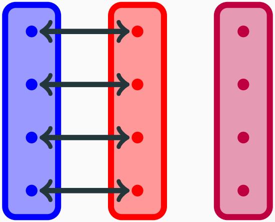
Purely Lacunary Systems

For purely lacunary supports $\mathcal{A}_\bullet$, there is an extension of the standard argument to determine $\mathcal{G}_{\mathcal{A}_\bullet}$.

- What happens if we lift a small loop around the discriminant?
  - A system in the discriminant now has many singular zeros.
  - Lifting a small loop does not produce a simple transposition.
- What is the analogue of 2-transitivity?
  - $\mathcal{G}_{\mathcal{A}_\bullet}$ is imprimitive and thus not 2-transitive.

Purely Lacunary Systems

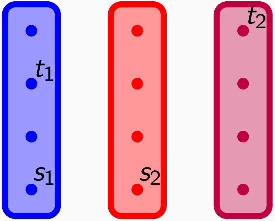
Proposition
Let $\mathcal{A}_\bullet$ be purely lacunary with reduced support $\mathcal{B}_\bullet$ such that $\mathcal{G}_{\mathcal{B}_\bullet}$ is a symmetric group. A small loop around the ($\mathcal{B}_\bullet$-)discriminant lifts to a simple permutation <u>of blocks</u> in $\mathcal{G}_{\mathcal{A}_\bullet}$.



- A general system $F \in \mathcal{P}_{\mathcal{A}_\bullet}$ in the ($\mathcal{B}_\bullet$-)discriminant has $|\mathcal{K}|$ singular zeros of multiplicity 2.

Purely Lacunary Systems

An imprimitive group $\mathcal{G}$ acting on a set S is <u>k-block-transitive</u> if for every $s_1, \ldots, s_k \in S$ from distinct blocks and $t_1, \ldots, t_k \in S$ from distinct blocks, there exists $g \in \mathcal{G}$ such that $g \cdot s_i = t_i$ for all $i = 1, \ldots, k$.



- 1-block-transitivity is equivalent to transitivity.
- 2-block-transitivity is strictly weaker than 2-transitivity.

Purely Lacunary Systems

Proposition
If $\mathcal{A}_\bullet$ is purely lacunary, then $\mathcal{G}_{\mathcal{A}_\bullet}$ is 2-block-transitive with respect to the blocks induced by its reduced support.

- (Block-)Transitivity properties of $\mathcal{G}_{\mathcal{A}_\bullet}$ are encoded in fiber powers of the projection $\pi : \mathcal{I}_{\mathcal{A}_\bullet} \to \mathcal{P}_{\mathcal{A}_\bullet}$.
- The result above follows by showing irreducibility of a certain component of the fiber square of the projection $\pi : \mathcal{I}_{\mathcal{A}_\bullet} \to \mathcal{P}_{\mathcal{A}_\bullet}$.

Purely Lacunary Systems

- Any element of $\mathcal{K} \wr \mathcal{S}_m$ can be represented in the form $(k_1, \ldots, k_m, \sigma) \in \mathcal{K} \wr \mathcal{S}_m$ for $k_1, \ldots, k_m \in \mathcal{K}$ and $\sigma \in \mathcal{S}_m$.

Define the map $\Sigma : \mathcal{K} \wr \mathcal{S}_m \to \mathcal{K}$ by

$$\Sigma((k_1, \ldots, k_m, \sigma)) = \prod_{i=1}^{m} k_i.$$

Proposition
If $\mathcal{G}$ is an imprimitive group which is 2-block-transitive and contains a simple transposition of blocks, then $\ker \Sigma \subseteq \mathcal{G}$.

Purely Lacunary Systems

Theorem (Y.)
If $\mathcal{A}_\bullet$ is purely lacunary, then the Galois group $\mathcal{G}_{\mathcal{A}_\bullet}$ contains the kernel of the map $\Sigma : \mathcal{K} \wr S_{\mathsf{MV}(\mathcal{B}_\bullet)} \to \mathcal{K}$. Thus, $\mathcal{G}_{\mathcal{A}_\bullet} = \Sigma^{-1}(\mathcal{H})$ for some subgroup $\mathcal{H} \subseteq \mathcal{K}$.

- Do all groups of this form appear as Galois groups of purely lacunary sparse systems?
  - The conjecture is, no.
  - The condition $[\mathcal{H} : \mathcal{K}] \mid \mathsf{MV}(\mathcal{B}_\bullet)$ seems to be necessary.
- Given $\mathcal{A}_\bullet$, can we determine the subgroup $\mathcal{H} \subseteq \mathcal{K}$?
  - Yes, there is a geometric realization of the map Σ!
  - $\mathcal{H}$ is the group of units that preserve a certain variety.

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Purely Lacunary Systems

Let $\mathcal{J}_{\mathcal{A}_\bullet}$ be any irreducible component of the variety

$$\{(F, x_1, \ldots, x_{\mathrm{MV}(\mathcal{A}_\bullet)}) \in \mathcal{P}_{\mathcal{A}_\bullet} \times \mathcal{S}^{\mathrm{MV}(\mathcal{A}_\bullet)} : F(x_i) = 0 \ \text{ for } \ \forall i\}$$

that projects dominantly to $\mathcal{P}_{\mathcal{A}_\bullet}$ and is not contained in the diagonal.

- A deck transformation of $\mathcal{J}_{\mathcal{A}_\bullet} \to \mathcal{P}_{\mathcal{A}_\bullet}$ is a birational automorphism of $\mathcal{J}_{\mathcal{A}_\bullet}$ fixing the fibers of $\mathcal{J}_{\mathcal{A}_\bullet} \to \mathcal{P}_{\mathcal{A}_\bullet}$.

Proposition
The Galois group $\mathcal{G}_{\mathcal{A}_\bullet}$ is equal to the group of deck transformations of $\mathcal{J}_{\mathcal{A}_\bullet} \to \mathcal{P}_{\mathcal{A}_\bullet}$, as groups of permutations of the zeros of a general system $F \in \mathcal{P}_{\mathcal{A}_\bullet}$.

Purely Lacunary Systems

For each block $B_1, \ldots, B_m$, let j_i be such that the j_i-th zero of the base system F lies in B_i.

Consider the map $\Pi : \mathcal{J}_{\mathcal{A}_\bullet} \to \mathcal{P}_{\mathcal{A}_\bullet} \times \mathcal{S}$ defined by

$$\Pi(F, x_1, \ldots, x_{\mathrm{MV}(\mathcal{A}_\bullet)}) = \left(F, \prod_{i=1}^{m} x_{j_i} \right).$$

Proposition
The kernel $\ker \Sigma \subseteq \mathcal{G}_{\mathcal{A}_\bullet}$ is equal to the subgroup of deck transformations of $\mathcal{J}_{\mathcal{A}_\bullet} \to \operatorname{Im} \Pi$.

Corollary
The Galois group $\mathcal{G}_{\mathcal{A}_\bullet}$ is equal to $\Sigma^{-1}(\mathcal{H})$ where $\mathcal{H} \subseteq \mathcal{K}$ is group of deck transformations of $\operatorname{Im} \Pi \to \mathcal{P}_{\mathcal{A}_\bullet}$.

Purely Lacunary Systems

We can write down equations for $\operatorname{Im}\Pi$!

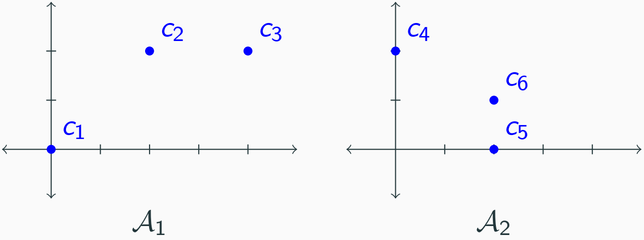
- Recall given $F \in \mathcal{P}_{\mathcal{A}_\bullet}$, there is $G \in \mathcal{P}_{\mathcal{B}_\bullet}$ such that $F = G \circ \phi$.
- The product of the zeros of G is a rational function in the coefficients of F, $p(F)$.

The variety $\operatorname{Im}\Pi \subseteq \mathcal{P}_{\mathcal{A}_\bullet} \times \mathcal{S}$ is a dominant, irreducible component of the variety defined by

$$\phi(x) - p(F) = 0.$$

Examples

Example: Consider the set of supports $\mathcal{A}_\bullet$ obtained by scaling the first coordinate of our example supports $\mathcal{B}_\bullet$. The coefficients of each monomial appears next to the corresponding point below.



The variety $\operatorname{Im}\Pi$ is an irreducible component of the variety defined by

$$c_3^2 c_5^2 x^2 - c_1^2 c_4^2 = 0$$
$$c_3 c_4^2 y - c_1 c_5^2 = 0.$$

Examples

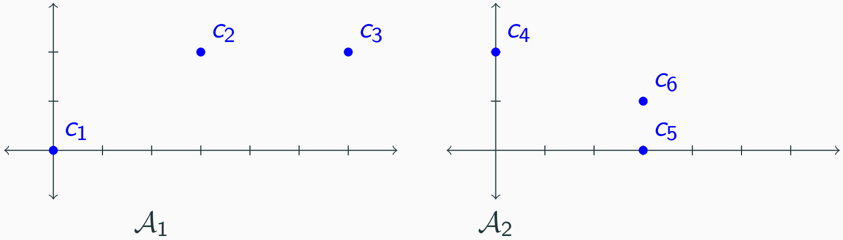
Example: This variety has two components that map dominantly to $\mathcal{P}_{\mathcal{A}_\bullet}$, defined by the systems below.

$$c_3 c_5 x - c_1 c_4 = 0 \qquad c_3 c_5 x + c_1 c_4 = 0$$
$$c_3 c_4^2 y - c_1 c_5^2 = 0 \qquad c_3 c_4^2 y - c_1 c_5^2 = 0$$

Both varieties have trivial group of deck transformations so that $\mathcal{G}_{\mathcal{A}_\bullet} = \Sigma^{-1}(\{e\}) = \ker \Sigma$, which has index 2 in the expected wreath product.

Examples

Example: If we had precomposed with the monomial map $\phi(x, y) = (x^3, y)$ to obtain the following support $\mathcal{A}_\bullet$:



The variety $\operatorname{Im}\Pi$ is the only dominant, irreducible component of the variety defined by

$$c_3^2 c_5^2 x^3 - c_1^2 c_4^2 = 0$$
$$c_3 c_4^2 y - c_1 c_5^2 = 0,$$

and $\mathcal{K} = \{(1,1), (\omega, 1), (\omega^2, 1)\}$ acts on it. Thus, $\mathcal{G}_{\mathcal{A}_\bullet}$ is the expected wreath product.

Some Remaining Questions

- Can this structure be exploited to reduce computation?
- Is there a conjecture for the "purely triangular" case?
- Is there any hope to determine the Galois group for systems which are both lacunary and triangular?

Thank You!

Thank you for your attention!

Other Events:

- Macaulay2 Workshop - Madison 2025. June 30-July 4, 2025.
- SIAM-AG 2025 - Madison. June 7-11, 2025.