Computing Galois groups of Fano problems

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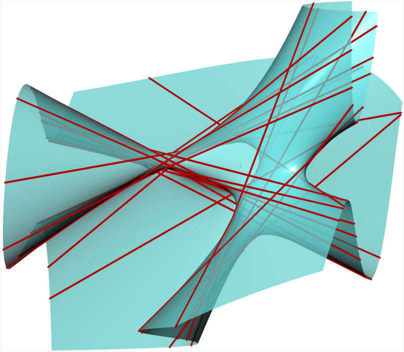
Lines in $\mathbb{P}^3$ on a cubic surface

- Cayley and Salmon showed there are 27 distinct lines that lie on a smooth cubic surface.
- Schläfli determined these lines lie in a "remarkable configuration"



Remark: In the image above, all 27 lines are real!

Fano problems

- The Grassmanian $\mathbb{G}(r, \mathbb{P}^n)$ is the space of r–planes in $\mathbb{P}^n$, a projective variety of dimension $(r+1)(n-r)$.
- The Fano scheme of $X \subseteq \mathbb{P}^n$ is the subscheme of the Grassmanian formed by the r–planes that lie on X.
- Example: The Fano scheme of lines in $\mathbb{P}^3$ on a smooth cubic surface consists of 27 points in $\mathbb{G}(1, \mathbb{P}^3)$.
- We consider Fano schemes of general complete intersections.

Fano Problems

A complete intersection $X \subseteq \mathbb{P}^n$ of codimension s is defined by polynomials $F = (f_1, \ldots, f_s)$ of some degrees $d_\bullet = (d_1, \ldots, d_s)$.

- A Fano scheme is classified by the data $(r, n, d_\bullet)$ called its type.
- Denote the space of systems F by $\mathbb{C}^{(r,n,d_\bullet)}$.
- The Fano scheme of r–planes on the zero set $X = \mathcal{V}(F)$ of $F \in \mathbb{C}^{(r,n,d_\bullet)}$ is written $\mathcal{V}_r(F)$.
- Example: The Fano scheme of lines in $\mathbb{P}^3$ on a cubic surface has type $(1, 3, (3))$.

Fano problems

The expected dimension of a Fano scheme of type $(r, n, d_\bullet)$ is

$$\delta(r, n, d_\bullet) = (r+1)(n-r) - \sum_{i=1}^{s} \binom{d_i + r}{r}.$$

Theorem (Debarre,Manivel)
A general Fano scheme of type $(r, n, d_\bullet)$ has dimension $\delta(r, n, d_\bullet)$ if $\delta(r, n, d_\bullet) \geq 0$ and $2r \leq n - s$, and is empty otherwise.

A Fano problem is a tuple $(r, n, d_\bullet)$ such that a general Fano scheme of type $(r, n, d_\bullet)$ is finite.

Example: The tuple $(1, 3, (3))$ is a Fano problem.

Examples

For a Fano problem $(r, n, d_\bullet)$, the general Fano scheme has a fixed cardinality called the degree of the Fano problem, $\deg(r, n, d_\bullet)$.

Debarre and Manivel give explicit formulas for this degree using techniques from intersection theory.

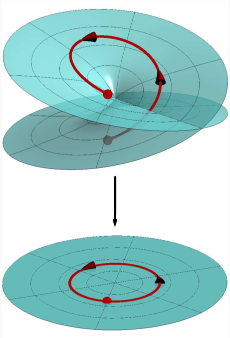
r	n	$d_\bullet$	$\deg(r, n, d_\bullet)$
1	4	$(2, 2)$	16
1	3	(3)	27
2	6	$(2, 2)$	64
3	8	$(2, 2)$	256
1	7	$(2, 2, 2, 2)$	512
1	6	$(2, 2, 3)$	720

Galois groups of Fano problems

There is an incidence correspondence.

$$\begin{array}{c} \Gamma = \{(F,\ell) \in \mathbb{C}^{(r,n,d_\bullet)} \times \mathbb{G}(r,\mathbb{P}^n) : \ell \in \mathcal{V}_r(F)\} \\ \quad \downarrow \pi_{(r,n,d_\bullet)} \\ \mathbb{C}^{(r,n,d_\bullet)} \end{array}$$

- The Galois group $\mathcal{G}_{(r,n,d_\bullet)}$, of the Fano problem $(r,n,d_\bullet)$ is the monodromy group of $\pi_{(r,n,d_\bullet)}$.
- Called "Galois groups" since Jordan defined them algebraically!



Galois groups of Fano problems

A complete classification of Galois groups of Fano problems is close!

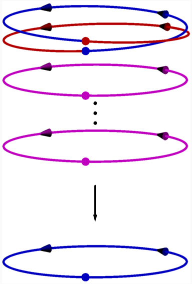
- $\mathcal{G}_{(1,3,(3))} = E_6$ [Jordan],[Harris].
- $\mathcal{G}_{(r,2r+2,(2,2))} = D_{2r+3}$ for $r \geq 1$ [Hashimoto,Kadets].
- If $(r, n, d_\bullet)$ is a Fano problem not equal to $(1, 3, (3))$ or $(r, 2r+2, (2, 2))$ for $r \geq 1$, then $\mathcal{G}_{(r,n,d_\bullet)}$ contains the alternating group [Hashimoto,Kadets].

Goal: Prove that for Fano problems not equal to $(1, 3, (3))$ or $(r, 2r+2, (2, 2))$ for $r \geq 1$, the Galois group is the symmetric group.

Harris' trick

To show some Galois groups contain a simple transposition, Harris exhibited a system F such that:

1. $\mathcal{V}_r(F)$ contains a unique double point.
2. $\mathcal{V}_r(F)$ contains $\deg(1, n, (2n-3)) - 2$ smooth points.



More specific plan: Find such systems for other Fano problems using computational tools.

Harris' trick

We prescribe a subscheme of $\mathcal{V}_r(F)$ to choose $F \in \mathbb{C}^{(r,n,d_\bullet)}$.

- Fix $\ell \in \mathbb{G}(r, \mathbb{P}^n)$ to lie in $\mathcal{V}_r(F)$.
- Fix a tangent vector at $\ell \in \mathcal{V}_r(F)$, $v \in T_\ell \mathcal{V}_r(F)$.

Choose F general satisfying these properties.

A point $x \in \mathbb{C}^m$ is a simple double zero of a square polynomial system G if $G(x) = 0$, $\ker DG(x) = \langle v \rangle$ for $v \neq 0$, and

$$D^2 G(x)(v, v) \notin \operatorname{im} DG(x).$$

Simple double zeros are isolated zeros of multiplicity 2.

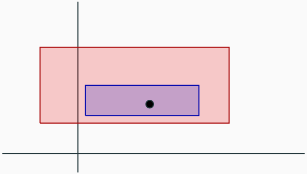
Harris' trick

The Krawczyk operator $K_{G,x,Y}$ acts on the space of complex intervals, given a square system G, $x \in \mathbb{C}^m$, and $Y \in \mathrm{GL}_m(\mathbb{C})$.

Theorem (Krawczyk)
If G, x, Y, and I are such that

$$K_{G,x,Y}(I) \subseteq I,$$

then I contains a zero of G.



- `HomotopyContinuation.jl` can find such complex intervals!

Results

Theorem (Y.)
The Fano problems not equal to $(1, 3, (3))$ or $(r, 2r + 2, (2, 2))$ for $r \geq 1$ and with less than 75,000 solutions have Galois group equal to the symmetric group.

- This determines the Galois group of 12 Fano problems which were previously unknown.

Data and code verifying this result is available at:

`github.com/tjyahl/FanoGaloisGroups`

Results

Timings are reported for verifying the simple double point and certifying the remaining solutions is given below.

r	n	$d_\bullet$	$\deg(r, n, d_\bullet)$	HomCo (s)
1	7	$(2, 2, 2, 2)$	512	.61
1	6	$(2, 2, 3)$	720	.87
2	8	$(2, 2, 2)$	1024	1.57
1	5	(3,3)	1053	.32
1	5	(2,4)	1280	.73
1	10	(2,2,2,2,2,2)	20480	15.44
1	9	(2,2,2,2,3)	27648	25.97
2	10	(2,2,2,2)	32768	36.67

Moving forward

There is more to do!

- (In progress) Generate and verify data for larger Fano problems. Current bottlenecks are memory and time!
- Turn this into a proof for ALL Fano problems not equal to $(1, 3, (3))$ or $(r, 2r+2, (2, 2))$ for $r \geq 1$.
- Explore using numerical certification to prove more about Galois groups and beyond.

Thank you all for your time!

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r	n	$d_\bullet$	$\deg(r, n, d_\bullet)$	`HomCo` (s)
1	7	$(2, 2, 2, 2)$	512	.61
1	6	$(2, 2, 3)$	720	.87
2	8	$(2, 2, 2)$	1024	1.57
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1	10	(2,2,2,2,2,2)	20480	15.44
1	9	(2,2,2,2,3)	27648	25.97
2	10	(2,2,2,2)	32768	36.67
1	8	(2,2,3,3)	37584	38.23
1	8	(2,2,2,4)	47104	111.88
1	7	(3,3,3)	51759	42.86
1	7	(2,3,4)	64512	125.63