Parameter Homotopies in Cox Coordinates

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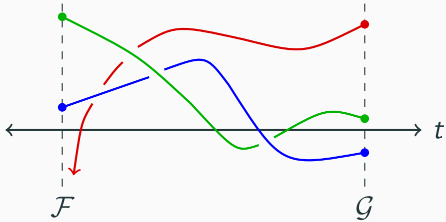
Numerical homotopy continuation

Uses numerical methods to "track" solutions from a start system $\mathcal{G}$ to a target system $\mathcal{F}$.



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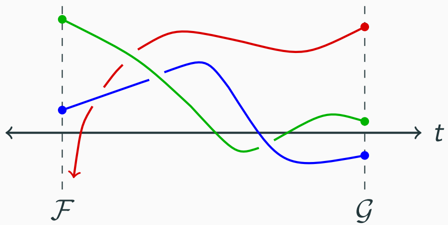
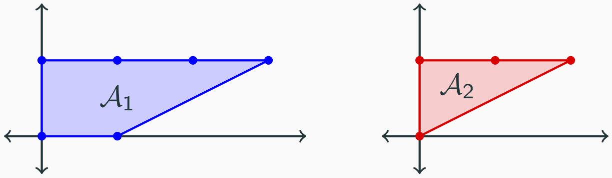


- Tracking can be done in (multi-)projective space to accomodate for divergent paths.

Sparse polynomial systems

Sparse polynomial systems are n polynomials in n variables whose monomials for each polynomial are predetermined.

Example: $\mathcal{F}(t_1, t_2) = \begin{pmatrix} 1 + t_1 + t_2 + t_1 t_2 + t_1^2 t_2 + t_1^3 t_2 \\ 1 + t_2 + t_1 t_2 + t_1^2 t_2 \end{pmatrix}$



We say $\mathcal{F}$ has support $\mathcal{A}_\bullet = (\mathcal{A}_1, \mathcal{A}_2)$.

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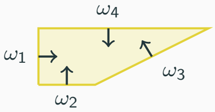
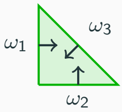
- We can understand this deficiency by compactifying our solution space.

Toric Varieties

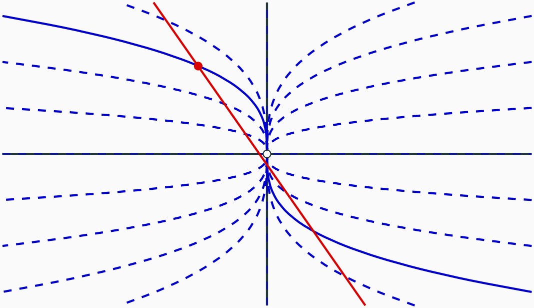
Given a polytope Δ, we can construct a toric variety X_Δ.

	X_Δ	$\mathbb{P}^2$
Polytope:		
Total Space:	$\mathbb{C}^4$	$\mathbb{C}^3$
Base Locus B:	$V(x_1, x_3) \cup V(x_2, x_4)$	$V(x_1, x_2, x_3) = \{0\}$
Group G:	$(\mathbb{C}^\times)^2$	$\mathbb{C}^\times$

The coordinates on X_Δ are called Cox coordinates. Points of $(\mathbb{C}^\times)^k$ embed as points with non-zero Cox coordinates.

Cox Coordinate Representatives

Points in X_Δ correspond to orbits $G \cdot x$ for $x \in \mathbb{C}^k \setminus B$.



By choosing a general affine space $L \subseteq \mathbb{C}^k \setminus B$ (of dimension n) we may select 1 of degree many representatives.

Homogenization

- We can homogenize polynomials f to polynomials $\widehat{f}$ on a toric variety X_Δ.
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Example: The homogenization $\widehat{\mathcal{F}}(x_1, x_2, x_3, x_4)$ of our system $\mathcal{F}(t_1, t_2)$ has $\mathrm{MV}(\mathcal{A}_\bullet) = 3$ solutions, with Cox coordinates

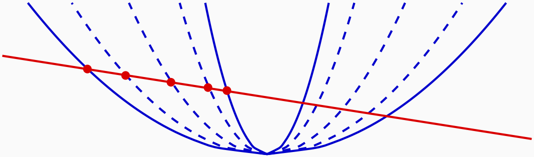
$$\{(-1,-1,1,1),(0,-1,1,1),(1,-1,0,1)\}.$$

The first solution $(-1,-1,1,1)$ to $\widehat{\mathcal{F}}$ corresponds to the unique solution $(-1,-1)$ to $\mathcal{F}$ in $(\mathbb{C}^\times)^2$.

Path Tracking and Endgame

From a start system $\mathcal{G}$ with the same support $\mathcal{A}_\bullet$ as $\mathcal{F}$, homogenize to obtain a system $\widehat{\mathcal{G}}$ and its MV($\mathcal{A}_\bullet$) solutions in X_Δ.

After choosing an affine space L and representativs of the solutions to $\widehat{\mathcal{G}}$, apply path-tracking methods!



Some representatives may diverge (or move into the base locus B). Endgame switches representatives as needed.

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1. Compute solutions to a start system $\mathcal{G}$ with the same support.
2. Homogenize to a system $\widehat{\mathcal{G}}$ on X_Δ.
3. Choose a generic affine space and compute representatives of the start solutions.
4. Use numerical homotopy methods to track start solutions to solutions of $\widehat{\mathcal{F}}$.

Thank You!

Thank you everyone for listening!

And thank you to the organizers!